Turbulent Diffusion: a transport matrix approach

Matrix Practice

A first look at vectors.

<2,3> and <1,6> are two vectors

Their scalar product is 2*1+3*6=20

Any two vector $\mathbf{A}=<x_1,y_1>$ and $\mathbf{B}=<x_2,y_2>$ have a scalar product (dot product) equal to

$$\vec{A} \cdot \vec{B} = x_1 x_2 + y_1 y_2$$

Question 1: What is the scalar product of <5,2> and <7,10> ?________________

What is the scalar product of <-1,8> and <-3,1> ?________________

Matrices:

A matrix is a two dimensional array made up of two row vectors (and two column vectors)

$A=\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ The two rows are <2,1> and <5,4> and the two columns are $\begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{and} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\begin{array}{ccc} & \text{column 1} & \text{column2} \\ \begin{matrix}\text{row 1}\\ \text{row2}\end{matrix} & \multicolumn{2}{c}{\begin{bmatrix}2 & 1\\ 5 & 4\end{bmatrix}} \end{array}$$

If the row and column entries are specified as

$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ the subscript 11 can be read first row and first column, subscript 12 1st row second column, subscript 21 second row first column, and subscript 22 second row second column.

The determinant of A is equal to $a_{11}(a_{22})-a_{21}(a_{12})$

Question 2: What is the determinant of $A=\begin{bmatrix}2 & 1\\ 5 & 4\end{bmatrix}$? Det A=_______________

Of $B=\begin{bmatrix}6 & 8\\ 3 & 4\end{bmatrix}$ Det B= _____________

If the determinant is zero the matrix is said to be singular and does not have a multiplicative inverse.

Matrix Multiplication.

Two matrices A and B

$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ and $B=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$

can be multiplied giving a new Matrix $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

c11 is the dot product of the first row vector with the first column vector

c12 is the dot product of the first row vector with the second column vector

c21 is the dot product of the second row vector with the first column vector

c22 is the dot product of the second row vector with the second column vector

Question 3: If $A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ what is C=AxB?

$$C = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

what is D=BxA?

$$D = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

(whenever a 2x2 matrix is multiplied by another 2x2 matrix the resulting product is a 2x2 matrix)

If $A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ what is M=AxI?

$$M = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

The Matrix $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ times a A column vector $\mathrm{V} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is another column vector U.

$\mathrm{U} = \mathrm{B} \times \mathrm{V} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ The top entry is the scalar product of the first row of B and V, and the bottom entry is scalar product of the second row of B with V.

Question 4: What is $\mathrm{U} = \mathrm{B} \times \mathrm{V} = \begin{bmatrix} \quad \end{bmatrix}$ if $B = \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathrm{V} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$?

Matrices can be useful for a variety of calculations. **Appendix 1** shows an example that may be familiar to the reader, in which matrices are used to solve simultaneous equations.

Pulling it together with an example of simple turbulent diffusion.

Review:

Exponential grow(decay). Remember we interpret the symbol dX/dt as the net flow into (out of) a stock X. when

$$\frac{dX}{dt} = rX \qquad then \quad X = X_0 e^{rt}$$

If r=0 then $X=X_0$ (net flow equals zero so the stock is in equilibrium)

If r>0 thev we have exponential growth

If r<0 then we have exponential decay.

Example 1. Consider two reservoirs A and B each of which has a content C_A and C_B. The transfer time controls the flow between A and B and we will use TT as an abbreviation here. If the two reservoirs are tanks containing fluid, the transfer time may depend on such physical parameters as the viscosity of the fluid flowing between the two tanks or the diameter of the pipe connecting the tanks. If the reservoirs are the northern hemisphere and southern hemisphere atmosphere, and the content is the build up of CFCs, then the transfer rate is related to how fast the air current mix air between the two hemispheres.

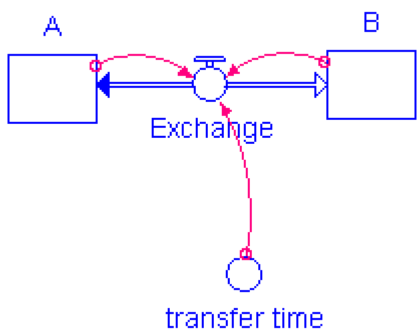


Figure. 1

The outflow from region A is $(C_A\text{-}C_B)/TT$. The net flow into A (inflow-outflow) is $(C_B\text{-}C_A)/TT = -C_A/TT + C_B/TT$

(This model for turbulent diffusion assumes an average transfer time, and that turbulent diffusion stops when the concentrations in A and B are equal)

Likewise, the Net flow into B is $C_A/TT\text{-}C_B/TT$

We can rewrite these net flow equations as

$$\frac{dC_A}{dt} = \frac{-C_A}{TT} + \frac{C_B}{TT}$$

$$\frac{dC_B}{dt} = \frac{C_A}{TT} - \frac{C_B}{TT}$$

Where the notation $\frac{dC_A}{dt}$ stands for net flow into A.

Or in vector/Matrix form

$$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \end{bmatrix} = \begin{bmatrix} -1/TT & 1/TT \\ 1/TT & -1/TT \end{bmatrix}\begin{bmatrix} C_A \\ C_B \end{bmatrix}$$

Or compactly as

$$\frac{dC}{dt} = KC \qquad \textbf{Equation 1}$$

where K is the transport matrix $K = \begin{bmatrix} -1/TT & 1/TT \\ 1/TT & -1/TT \end{bmatrix} = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$

And C is the column vector representing the contents of each reservoir

$$C = \begin{bmatrix} C_A \\ C_B \end{bmatrix}$$

Sometimes scientists like to write equations in matrix form, like **equation 1**, for compactness.

Question 5:

If $K = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} C_A \\ C_B \end{bmatrix}$, what are the two flow equations implied by $\frac{dC}{dt} = KC$?

$$\frac{dC_A}{dt} = \_\_\_\_\_ C_A + \_\_\_\_\_\_ C_B$$

$$\frac{dC_B}{dt} = \_\_\_\_\_ C_A + \_\_\_\_\_\_ C_B$$

If $K = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} C_A \\ C_B \end{bmatrix}$, what are the two flow equations implied by $\frac{dC}{dt} = KC$?

The solution to Equation 1 is exponential in character. That is

$$C_A = A1 * e^{r1*time} + A2 * e^{r2*time}$$
$$C_B = B1 * e^{r1*time} + B2 * e^{r2*time}$$

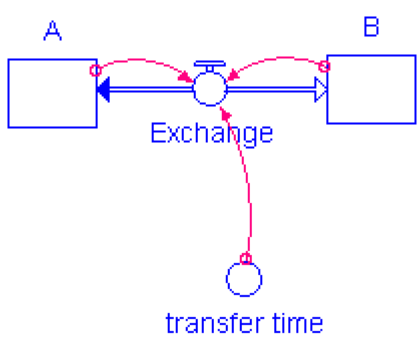
Where r_1 and r_2 are the eigenvalues of the transport matrix K and A_1, A_2, B_1, B_2 depend on the initial values of C_A and C_B.

We will not derive this highlighted assertion but take this as a given. Negative eigenvalues correspond to exponential decay, positive values correspond to exponential growth, and an eigenvalue of zero correspond to a constant solution.

Calculating Eigenvalues can be tedious. Lucky for us there are calculators that do this easily. **Appendix 2** shows an example of how to calculate eigenvalues for a 2x2 matrix

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

To Summarize.



The above diffusion model can be expressed in equation form as

$$\frac{dC}{dt} = KC \qquad \textbf{Equation 1}$$

where K is the transport matrix $K = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$, k=.05

And C is the column vector representing the contents of each reservoir $C = \begin{bmatrix} C_A \\ C_B \end{bmatrix}$

The solution for C_A and C_B is the sum of two exponential terms

$$C_A = A1 * e^{r1*time} + A2 * e^{r2*time}$$
$$C_B = B1 * e^{r1*time} + B2 * e^{r2*time}$$

Where r_1 and r_2 are the eigenvalues of the transport matrix K and A_1, A_2, B_1, B_2 depend on the initial values of C_A and C_B. The Eigenvector describe how CA and CB are partitioned for each eigenvalue. For example,

When $K = \begin{bmatrix} -0.05 & 0.05 \\ 0.05 & -0.05 \end{bmatrix}$

The eigenvalues are **r1=0 and r2=-.1** and the eigenvectors are <1,1> and <1,-1> respectively.

so

$$C_A = A + B * e^{-0.1*time}$$
$$C_B = A - B * e^{-0.1*time}$$

The r1=0 eigenvalue is shared equally between C_A ad C_B, and the r2=-0.1 eigenvalue is equal but has the opposite sign in C_A compared to C_B.

Since at t=0 C_A=100 A+B=100 and since at t=0 C_B=0 A=B

This gives A=B=50

So the final solution is.

$$C_A = 50 + 50 * e^{-0.1*time}$$

$$C_B = 50 - 50 * e^{-0.1*time}$$

The zero value of rate constant (eigenvalue) corresponds to a constant fixed level and the -0.1 yr^{-1} corresponds to a goal seeking behavior with a time constant of 10 years. The behavior of the two reservoir system with initial values for C_A=100 and C_B=0 is shown below. Notice the gap between A and B reduces by 37 % in about 10 years (the half gap time is 0.7*10 years= 7.0 years).

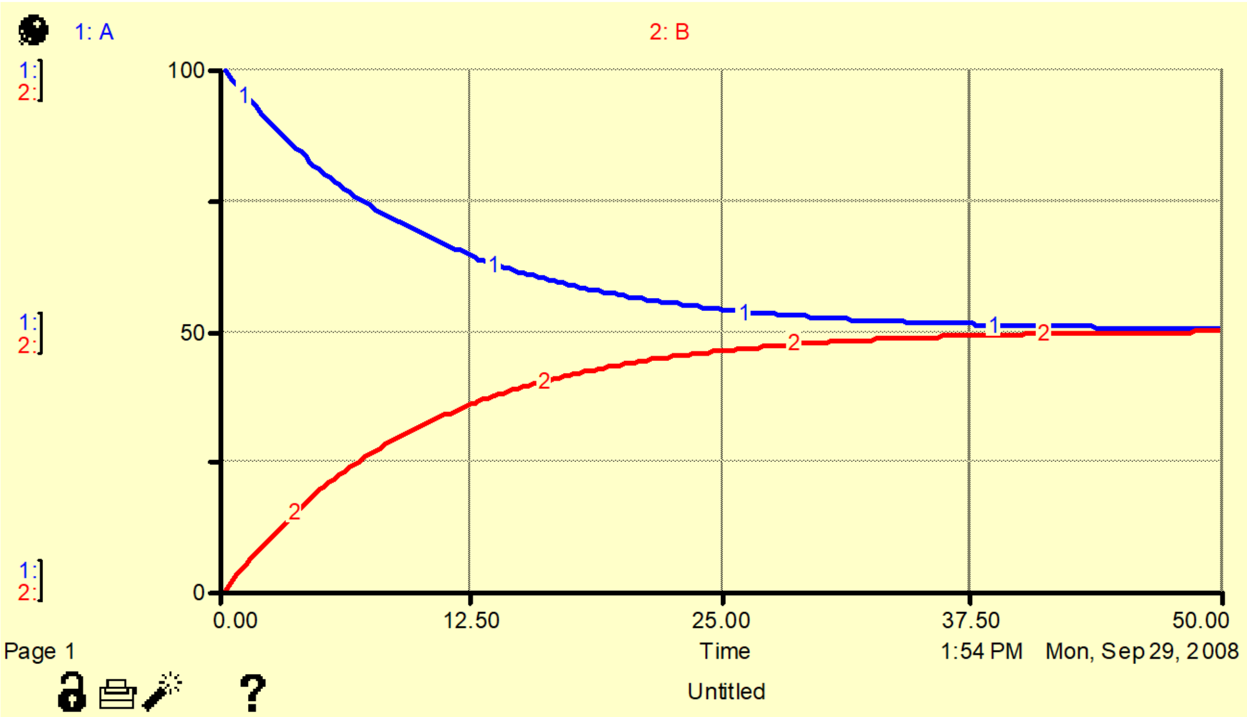


Figure 2. Response of a 2-reservoir system with no outflow only exchange between the two reservoirs.

Question 6: For the two reservoir diffusion model described above what would be the eigenvalues when the transport time TT=40 years?

r1=_______________

r2=________________

If the initial values of C_A and C_B are 100 and 20 respectively, what will be the final equilibrium content of each?

Final equilibrium = ____________________

Sketch the behavior of this system (TT=40 years) on the axes below

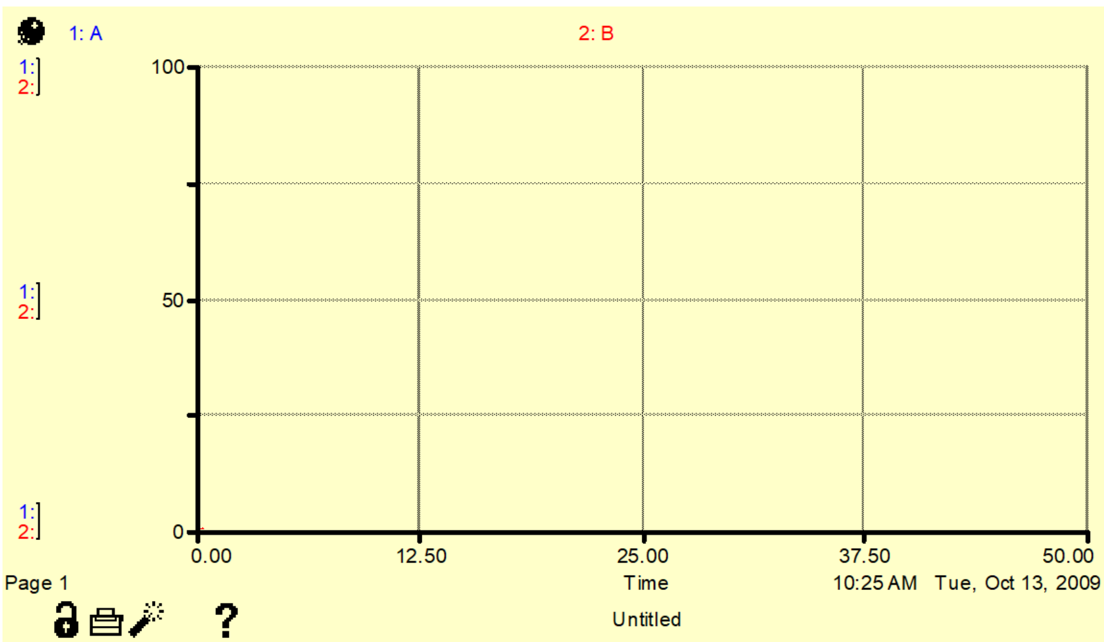


Figure 3.

A three reservoir system (A, B, and C) would have a 3X3 transport matrix and 3 eigenvalues (3 exponential terms),

$$C_A = A1 * e^{r1*time} + A2 * e^{r2*time} + A3 * e^{r3*time}$$

$$C_B = B1 * e^{r1*time} + B2 * e^{r2*time} + B3 * e^{r3*time}$$

$$C_C = C1 * e^{r1*time} + C2 * e^{r2*time} + C3 * e^{r3*time}$$

a four reservoir system would have a 4X4 transport matrix and 4 eigenvalues (4 exponential terms), …

Question 7: How many exponential terms are needed to completely describe the behavior of a 5 reservoir system?

Four points are worth noting here about the eigenvalues, r:

1) When r=0 the exponential term is a constant term,
2) when r is negative the exponential term eventually goes to zero
3) if r is positive there will be exponential growth.
4) If r is a complex number this corresponds to oscillations

Example 2.

For this example, the dynamic behavior of a two reservoir system is described by

$\frac{dC}{dt} = KC$ with $K = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$ The Stella model structure is shown below.

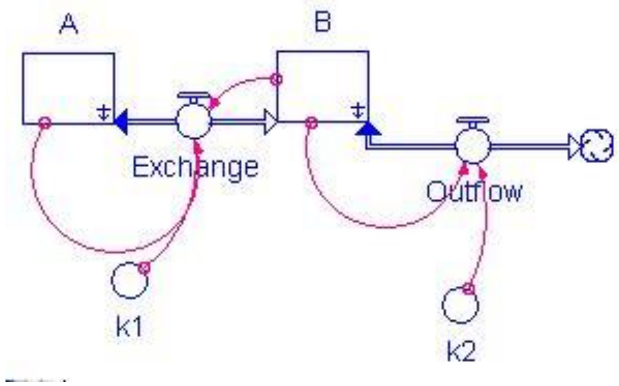


Figure 5.

A(t) = A(t - dt) + (- Exchange) * dt
INIT A = 100
OUTFLOWS:
Exchange = (A-B)*k1
B(t) = B(t - dt) + (Exchange - Outflow) * dt
INIT B = 0
INFLOWS:
Exchange = (A-B)*k1
OUTFLOWS:
Outflow = B*k2
k1 = 1
k2 = 1

The Calculator for Eigenvalues and Eigenvectors gives the values below for the transport matrix K.

Real eigenvalues:
{-2.618033988749895, -0.3819660112501051} **= { -2.62 , -0.38 }**
Eigenvector of eigenvalue -2.618033988749895:
(-0.5257311121191336, 0.8506508083520399) **= (-0.526, 0.851)**
Eigenvector of eigenvalue -0.3819660112501051:
(0.8506508083520399, 0.5257311121191336) **= (0.851, 0.526)**

Question 8:

1. What are the time constants of this 2-reservoir system?
2. What is the final equilibrium value of each reservoir?
3. What is the ratio of the content in reservoir A to that of B after 6 years?

Question 8b: What is the analytic solution for this 2 reservoir system?

Example 3. As another example let's say that the concentration (atoms/cc) in each reservoir is the true driver for diffusion and that reservoir A is twice as large (by volume) as B. In this case the outflow of A will go into B, but the increase in the concentration of B is twice that of the decrease in concentration of A. The transport matrix will now look like

$$K = \begin{bmatrix} -1/TT & 1/TT \\ 2/TT & -2/TT \end{bmatrix}$$

If the transport time TT=20 years, then the transport matrix $K = \begin{bmatrix} -0.05 & 0.05 \\ .1 & -0.1 \end{bmatrix}$

Solving for the eigenvalues either by hand or using the calculator gives

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm

r1=0 and r2=-0.15

The time constants controlling the behavior of the two reservoir system are t1=1/r1 and t2=1/r2. Large negative eigenvalues correspond to rapidly decaying exponential terms and smallest negative eigenvalue (or the largest positive eigenvalue) correspond to the dominant long term behavior. This eigenvalue is also sometimes referred to as the dominate eigenvalue since its behavior persists.

The **eigenvectors**, corresponding to each eigenvalue, describe the partitioning of the mass (or material) to each reservoir.

Using an initial concentration of <100,0> the above transport matrix

$K = \begin{bmatrix} -0.05 & 0.05 \\ .1 & -0.1 \end{bmatrix}$ Yields

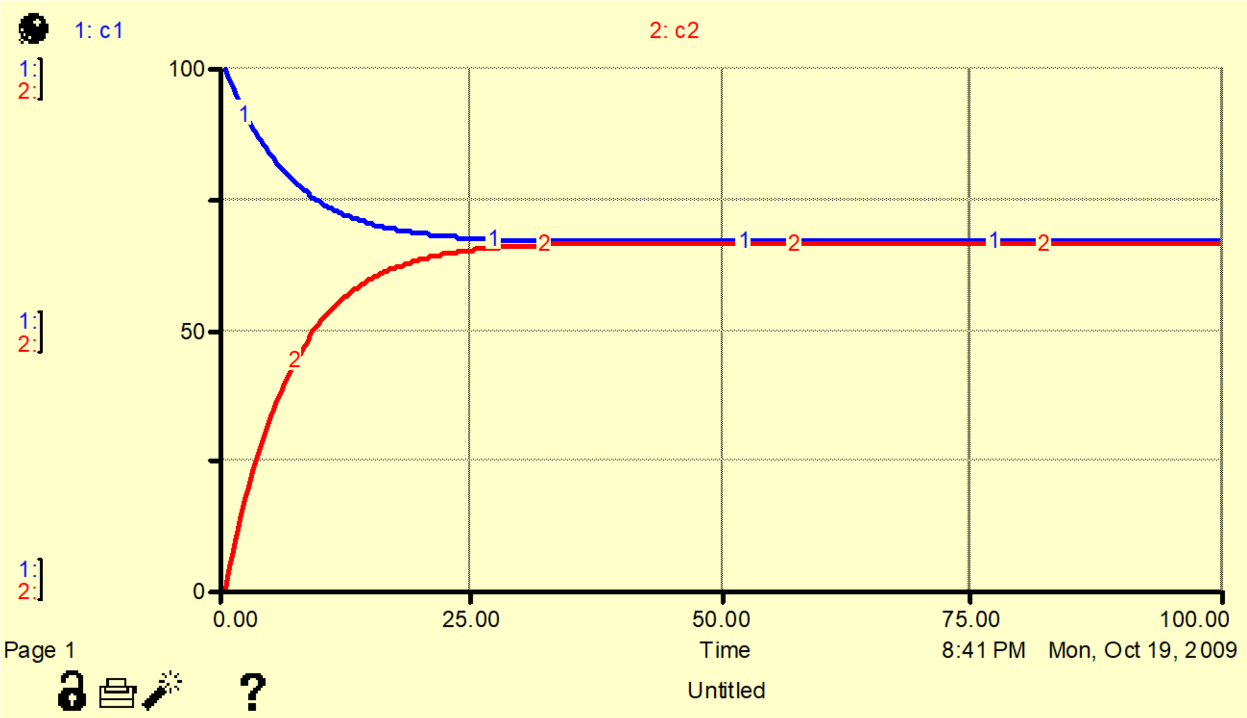


Figure 4.

The eigenvectors are <1,1> for r1=0 and <-1,2> for r2=-0.15.

The eigenvector <1,1> for r1=0 indicates that in the long term, the reservoirs have equal concentrations. The eigenvector <-1,2> for r2=-0.15 implies that in the early response reservoir 1 decreases half as much as reservoir 2 increases.

The two solutions for the concentration in each reservoir are:

C1=66.7+33.3*exp(-0.15t)

C2=66.7-66.7*exp(-0.15 t)

Formally we can obtain these solutions from

$$C_A = A1 * e^{r1*time} + A2 * e^{r2*time}$$
$$C_B = B1 * e^{r1*time} + B2 * e^{r2*time}$$

And the initial conditions, eigenvalues, and eigenvectors. Or for this specific case,

$$C_A = A + B * e^{-0.15t}$$

$$C_B = A + (-2B) * e^{-0.15t}$$

Here we've used the eigenvalues r1=0, r2=-0.15 and the eigenvectors <1,1> and <1,-2>.

At t=0 CA=100 and CB=0 so

A+B=100

A-2B=0

Solving for A and B A =66.7 and B=33.3 giving

$$C_A = 66.7 + 33.3 * e^{-0.15t}$$

$$C_B = 66.7 - (66.7) * e^{-0.15t}$$

Time Delay

When two flows come out of one stock the total effective rate constant is simply the sum of the separate rate constants Ktot=k1+k2. See Stella example below.

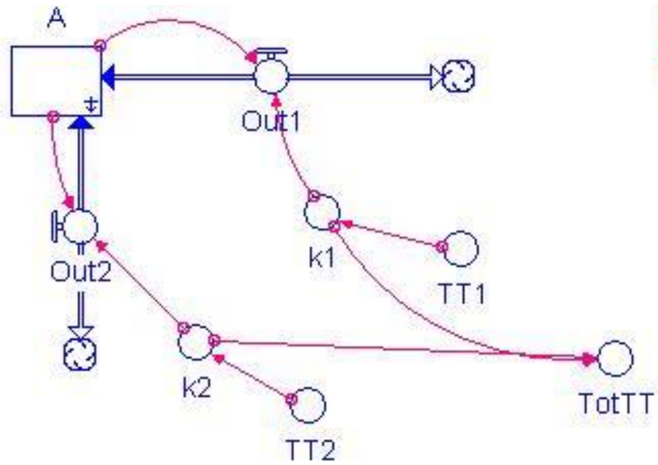


Figure 6.

A(t) = A(t - dt) + (- Out1 - Out2) * dt
INIT A = 100
OUTFLOWS:
Out1 = k1*A
Out2 = k2*A
k1 = 1/TT1
k2 = 1/TT2
TotTT = 1/(k1+k2)
TT1 = 10
TT2 = 20

The above stella equations set k1=0.1 yr^{-1}, k2=0.05 yr^{-1} and the total transit time for the system TotTT=1/(.1+.05)=6.67years

Figure 7.

If there are two stocks with two separate flows it is a bit more complicated.

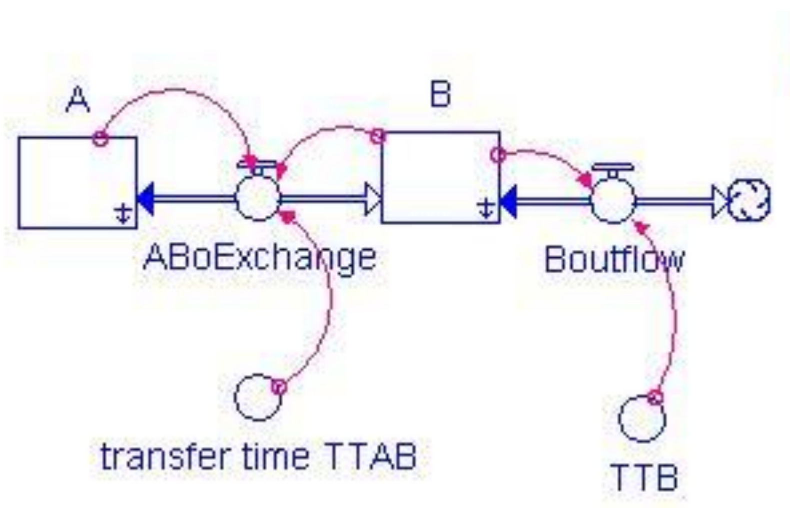


Figure 8.

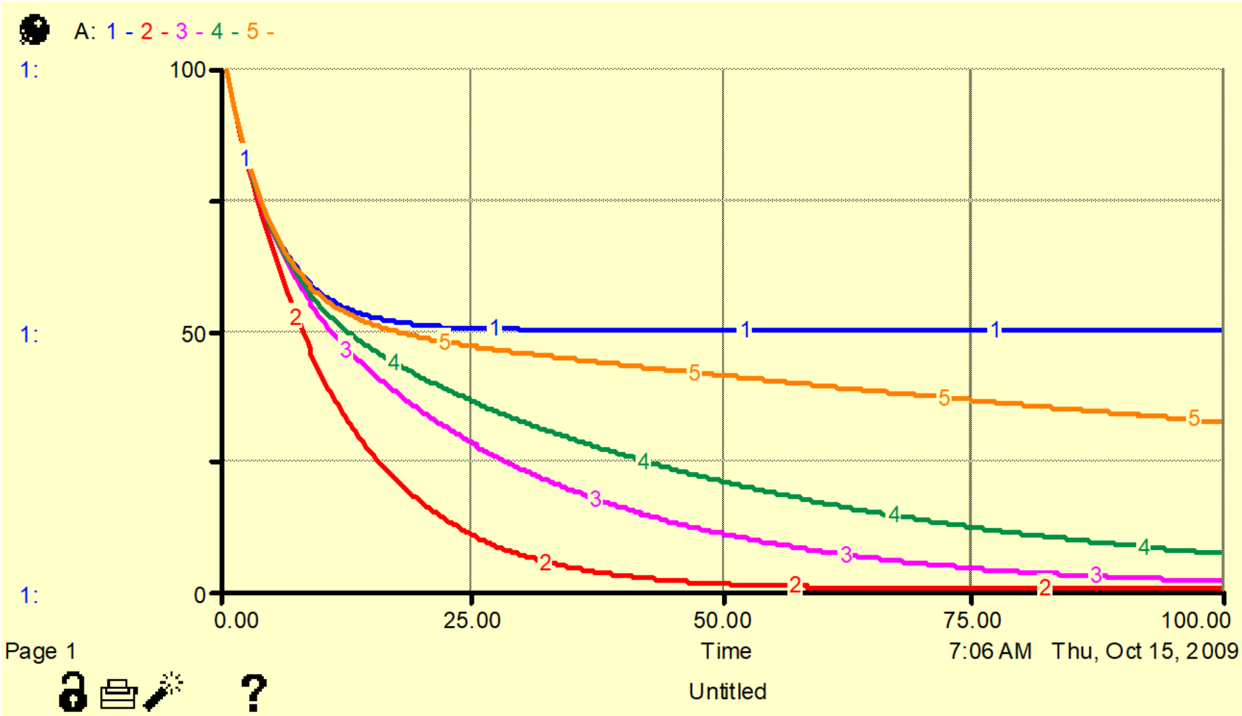


Figure 9.

The graph above shows 100 year runs of the two reservoir model. TTAB is held fixed at 10 years and TTB is change from $1\text{x}10^9$ (no outflow) , 100, 20, 10, 1 year.

With these conditions the transport Matrix is:

$$K = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -\left(0.1 + \frac{1}{TTB}\right) \end{bmatrix}$$

The eigenvalues (r1 and r2) for each run ($TTB=1\text{x}10^9$, 100, 20, 10, 1 year) are summarize in the table below. (Remember TTAB=10 years)

AB TT=10 yrs for all table entries. TTB is as shown in the table.

TTB	r1	r2
$1x10^9$	-0.2 yr^{-1} (5 year)	0
100	-0.205 yr^{-1} (5 year)	-.0049 yr^{-1} (200 year)
50	-.21 yr^{-1} (4.8 yr)	-.0095 yr^{-1} (105 yr)
20	-0.228 yr^{-1} (4.4 yrs)	-0.0219 yr^{-1} (45.7 yr)
10	-0.262 yr^{-1} (3.8 yr)	-0.038 yr^{-1} (26.3 yr)
1	-1.11 yr^{-1} (.9 year)	(-0.090 yr^{-1}) (11 yr)

Table 1.

When TTB=$1x10^9$ there is essentially no outflow from B and the system behaves as a pure exchange system with an equilibrium material content in each being equal to ½ the total initial material content of both. For an AB exchange rate of 0.1 yr^{-1} (10 yr AB Transfer time) the system response has a decay rate of 0.2 yr^{-1} or response time of 5 years. The response of this 2-reservoir system is identical to that shown in Figure 2.

When TTB=100 years the fast response is still about 5 years (half the AB transfer time of 10 years) but there is a slow response time of ~200 year representing loss of material from the system. Notice that this slow response is twice TTB. The same is true for TTB=50 years.

*Rule of thumb: This is generally true for this sytem structure shown in Figure 8. If TTB >> TTABThe fast response time is TTAB/2 and the slow response time is about 2*TTB. Another way of saying this is that the fast (largest eigenvalue) rate constant is about 2*k_{AB} and the slow long term rate constant (smallest eigenvalue) is about $k_{loss\ from\ B}$/2.*

When TTB=20 years the fast response is still about 5 years but there is a slow response time of ~46 years representing loss of material from the system. Notice that this slow response is just over twice TTB. This still roughly fits the rule of thumb described above despite the fact that TTB is only twice as large as TTAB.

Question 9: Assume that for a two reservoir system structure shown in Figure 8 the AB transfer time is 2 years (k_{AB}=0.5 yr^{-1}) and the loss lifetime from B (TTB) is 20 years ($k_{loss\ from\ B}$=0.05 yr^{-1}). . Estimate the two eigenvalues and their corresponding transfer times. What is the transfer time (or time constant) linked to the long term behavior of this system?

As a last example let's modify the model shown in Figure 8 slightly. We will use k_{AB}=0.1 yr^{-1} (10 yr) and $k_{B\ loss}$=0.02 yr^{-1} (50 yr) for our base model. With these value the transport matrix is

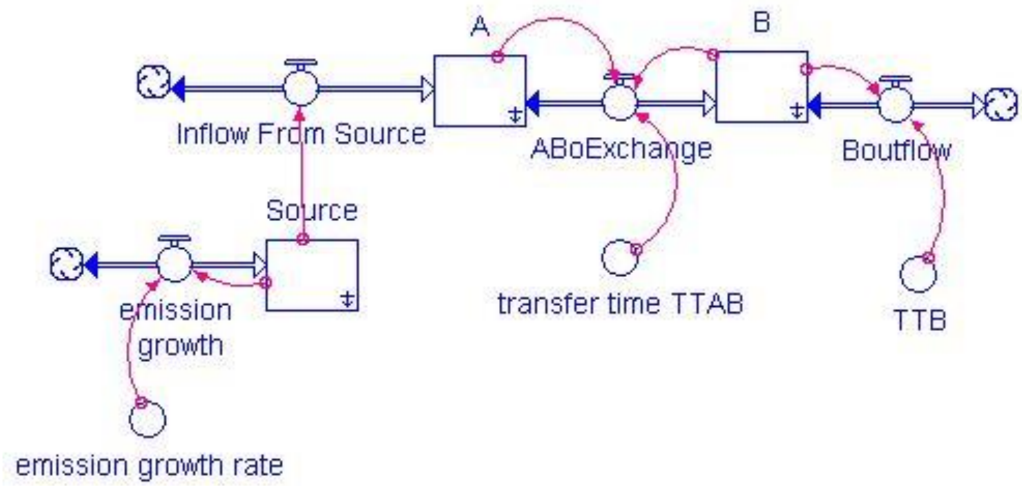
$$K = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -(0.12) \end{bmatrix}$$

This transport Matrix combined with

$\frac{dC}{dt} = KC$ where the content vector $C = \begin{bmatrix} A \\ B \end{bmatrix}$ imply that the flows for A and B are

$$\frac{dA}{dt} = -0.1A + 0.1B$$
$$\frac{dB}{dt} = 0.1A - 0.12B$$

Now lets add a new reservoir as a source of material as shown below



Stella Equations

A(t) = A(t - dt) + (Inflow_From_Source - ABoExchange) * dt

INIT A = 100

INFLOWS:

Inflow_From_Source = Source

OUTFLOWS:

ABoExchange = (A-B)/transfer_time_TTAB

B(t) = B(t - dt) + (ABoExchange - Boutflow) * dt

INIT B = 0

INFLOWS:

ABoExchange = (A-B)/transfer_time_TTAB

OUTFLOWS:

Boutflow = B/TTB

Source(t) = Source(t - dt) + (emission__growth) * dt

INIT Source = 2

INFLOWS:

emission__growth = emission_growth_rate*Source

emission_growth_rate = 0

transfer_time_TTAB = 10

TTB = 100

We first start with a fixed emission source of 2.0 tons of material per year so the emission growth rate r_{ems}=0.

With the added reservoir we're going to need a 3x3 matrix and an extra flow equation. The new flow equations are

$$\frac{dA}{dt} = -0.1A + 0.1B + 2.0$$

$$\frac{dB}{dt} = 0.1A - 0.12B + 0$$

$$\frac{dS}{dt} = 0.0A + 0.0B + 0$$

And the new transport matrix K is

$$K = \begin{bmatrix} -0.1 & 0.1 & 2.0 \\ 0.1 & -0.12 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The three eigenvalues of this transport matrix are:

{-0.21, -0.0095, 0} These are the same as row 3 table 1 with the addition of r=0

Answers to all questions

Question 1: What is the scalar product of <5,2> and <7,10> ? **55**

Question 2: What is the determinant of $A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$? **3 determinant of D is zero. If the determinant is zero the matrix is said to be singular and does not have a multiplicative inverse.**

Question 3: If $A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ what is C=AxB? $C = \begin{bmatrix} 7 & 2 \\ 19 & 8 \end{bmatrix}$

Question 4: What is $\mathrm{U} = \mathrm{B} \times \mathrm{V} = \begin{bmatrix} \quad \end{bmatrix}$ if $B = \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathrm{V} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$?

$$\mathrm{U} = \begin{bmatrix} 22 \\ 5 \end{bmatrix}$$

Question 5:

If $K = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$ what are the two flow equations implied by $\frac{d\mathrm{C}}{dt} = K\mathrm{C}$?

$$\boxed{\begin{aligned} \frac{d\mathrm{C_A}}{\mathrm{dt}} &= -2\mathrm{C_A} + \mathrm{C_B} \\ \frac{d\mathrm{C_B}}{\mathrm{dt}} &= \mathrm{C_A} - 4\mathrm{C_B} \end{aligned}}$$

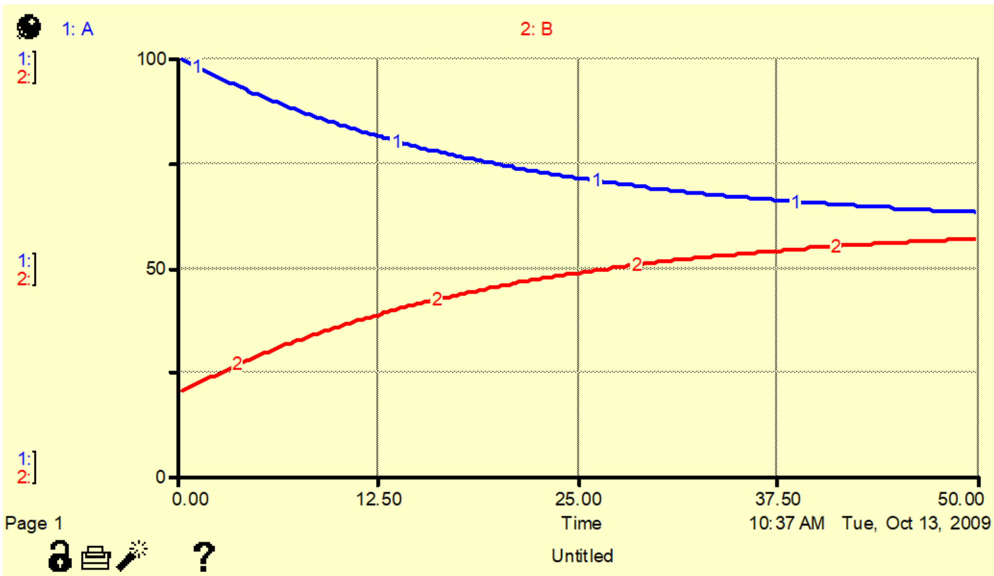
Question 6: For the two reservoir diffusion model described above what would be the eigenvalues when the transport time TT=40 years?

r1=**0.0**

r2**=-0.05**

If the initial values of C_A and C_B are 100 and 20 respectively, what will be the final equilibrium content of each? **60**

Question 7: How many exponential terms are needed to completely describe the behavior of a 5 reservoir system? **Five**



The dynamic behavior of a two reservoir system is described by

$$\frac{dC}{dt} = KC \quad \text{with} \quad K = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$ The Stella model structure is shown below.

Real eigenvalues:
{-2.618033988749895, -0.3819660112501051}
Eigenvector of eigenvalue -2.618033988749895:
(-0.5257311121191336, 0.8506508083520399)
Eigenvector of eigenvalue -0.3819660112501051:
(0.8506508083520399, 0.5257311121191336)

Question 8:

1. What are the time constants of this 2-reservoir system?

   $Time_1$=-1/-2.62=0.382 years $Time_2$=-1/-0.382=2.62 years

2. What is the final equilibrium value of each reservoir? **CA=CB=0.0**

(Both decay exponentially to 0.0, we need an eigenvalue of 0.0 for there to be a constant term indicating a leveling off to a non-zero equilibrium value)

3. What is the ratio of the content in reservoir A to that of B after 6 years?
After 6 years the fast response can be ignored and only the response of the smallest eigenvalue survives. Thus the eigen vector related to -0.382 eigenvalue gives the partion of material between the two reservoirs.

CA/CB=0.851/0.526=1.62

The fast response has a time constant of 0.38 years and after 5 time constants (1.9 yr) all effects are gone from the system.

Question 8b.

Real eigenvalues:

= { -2.62 , -0.38 }

Eigenvector of eigenvalue -2.62:

= (-0.526, 0.851)

Eigenvector of eigenvalue -0.38:

= (0.851, 0.526)

$$\mathrm{C_A} = A1 * e^{r1*time} + A2 * e^{r2*time}$$

$$\mathrm{C_B} = B1 * e^{r1*time} + B2 * e^{r2*time}$$

Or

$$\mathrm{C_A} = -.526A * e^{-2.62t} + .851B * e^{-0.38t}$$

$$\mathrm{C_B} = .851A * e^{-2.62t} + .526B * e^{-0.38t}$$

At t=0

100=-.526 A+.851B

0= .851 A +.526B solving for A and B A=52.6 and B=85.0

Giving

$$C_A = -27.6A * e^{-2.62t} + 72.4 * e^{-0.38t}$$

$$C_B = 44.7 * e^{-2.62t} + 44.7 * e^{-0.38t}$$

Question 9: For a two reservoir system structure shown in Figure 8 the AB transfer time is 2 years (k_{AB}=0.5 yr^{-1}) and the loss lifetime from B (TTB) is

20 years ($k_{loss\ from\ B}$=0.05 yr^{-1}). Estimate the two eigenvalues and their corresponding transfer times. What is the transfer time (or time constant) linked to the long term behavior of this system?

Compare with when AB TT= 10 yrand TTB=100 yr which gave 5 yr and 200 yr response times, (r1=.2 and r2=.005)

r1~1 yr^{-1} and r2~.025 yr^{-1}. These values correspond to response time of 1 yr and 40 years respectively. The 40 year response time represents the long term decay of the system to zero and is considered the dominant eigenvalue.

Appendix 1. Solving two equations with two unknowns, a matrix solution.

A quick example to highlight one use of matrices:

You may have seen the matrix solution to simultaneous equations in one of your past math classes.

Two equations with two unknowns are common in many environmental problems. Let say you have the two linear equations:

$2x + 4y = 6$

$3x - 2y = 1$

these can be written as a matrix time a column vector

$$\begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

setting

$$\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \vec{A} = \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}, \textit{and} \ \vec{C} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

we have

$$\vec{A}\vec{X} = \vec{C}$$

or

$$\vec{X} = \vec{A}^{-1}\vec{C} \qquad \text{where } \vec{A}^{-1}\vec{A} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1	0	=MMULT(Ainv,A)
0	1	

				CV=	
A=	2	4	X=	6	
	3	-2	Y=	1	
Ainv=	0.125	0.25	=MINVERSE(A)		
	0.1875	-0.125			
			X=	1	'=MMULT(Ainv,CV)
			Y=	1	

The highlighted area above is simply an example of one way that matrices are used and may be presented in the lab later. This example will not be on the exam.

Appendix 2. Solving for eigenvalues of a 2x2 transport matrix.

Optional: It's not too bad for our 2x2 matrix though, especially when it's of the symmetrical form as in the A and B reservoir system example. For this case

$$K = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$$

The way to do it is to find the values of r that make the determinant of **K-rI=0,**

Where I is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$K - rI = \begin{bmatrix} -k-r & k \\ k & -k-r \end{bmatrix}$ so the determinant is

$$[-(k+r)]^2 - k^2 = k^2 + 2kr + r^2 - k^2 = +2kr + r^2$$

Setting this equal to zero

$$2kr + r^2 = 0$$

Or $(2k + r)r = 0$

The values of r that make the left side equal to zero are r=0 and r=-2k.

Thus for a transport time of TT=20 years and k=1/TT=0.05 yr^{-1}. The transport Matrix is

$$K = \begin{bmatrix} -0.05 & 0.05 \\ 0.05 & -0.05 \end{bmatrix},$$

and the eigenvalues are 0 and -2(.05)=-0.1 yr^{-1}.

The information in the highlighted region is optional since we will normally use the calculator at the link below for the tough work.

http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm